CALCULATING INTERSECTION NUMBERS ON MODULI SPACES OF CURVES

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Abstract. We discuss an algorithm for calculating intersection numbers for tautological classes on $\overline{\mathcal{M}}_{g,n}$, and use this to compute the coefficients of a genus 4 tautological relation in cohomology whose existence follows directly from the work of Bergström-Tommasi. We end with the ranks of the graded parts of the Gorenstein quotients of the tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$, as well as of the related rings $R^*(\mathcal{M}_{g,n}^{\mathrm{ct}})$ and $R^*(\mathcal{M}_{g,n}^{\mathrm{rt}})$, for low values of g and n.

Contents

1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space of Deligne-Mumford stable curves of genus g with n labeled points. The tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ comprise the smallest system of $\mathbb{Q}$-subalgebras of the Chow rings $A^*(\overline{\mathcal{M}}_{g,n})$ that is closed under the natural forgetful and gluing morphisms. These rings include the cotangent classes ψ_i, the Mumford-Morita-Miller classes κ_i, the Chern classes λ_i of the Hodge bundle, and topologically-defined boundary classes (for definitions and properties, see [AC, HKK$^+$, M]).

Programs for computing top intersection numbers among ψ, κ, and λ classes have been implemented in both Maple and Macaulay2 [F3, SY]. However, neither of these handle boundary classes in a satisfactory manner. An algorithm for doing so is described below in terms of decorated stable graphs. This is based on a formula given a paper by Graber and Pandharipande ([GP, §A.1]) and has has been implemented in Maple. This program has been used to calculate the ranks of the Gorenstein quotients of various tautological rings of moduli spaces of curves $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}_{g,n}^{\mathrm{ct}}$, and $\mathcal{M}_{g,n}^{\mathrm{rt}}$ for low values of g and n, where $\mathcal{M}_{g,n}^{\mathrm{ct}}$ denote the locus of curves in $\overline{\mathcal{M}}_{g,n}$ "of compact type," whose dual graph is a tree, and $\mathcal{M}_{g,n}^{\mathrm{rt}}$ denotes the

locus of curves in $\overline{\mathcal{M}}_{g,n}$ "with rational tails," with one component of genus g. According to well-known conjectures ([F2, P]), the tautological rings of $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}^{\mathrm{ct}}_{g,n}$ and $\mathcal{M}^{\mathrm{rt}}_{g,n}$ are Gorenstein and thus equal to their Gorenstein quotients.

2. Stable graphs

Graber and Pandharipande were the first to write down an explicit multiplication formula for boundary classes in $R^*(\overline{\mathcal{M}}_{g,n})$ ([GP, §A.1]); we begin by adapting their notation. By a graph, we mean a connected and undirected graph with allowed half-edges, multiple edges, and self-edges. In other words, it is a sextuple

$$(V, H, E, N, g\colon V \to \mathbb{Z}_{\geq 0}, i\colon H \to H)$$

which satisfies the properties:

(1) V is a finite set of vertices, with a genus function $g\colon V \to \mathbb{Z}_{\geq 0}$.
(2) H is a finite set of half-edges, and i is a involution with labeled fixed points N.
(3) E is the set of nontrivial orbits of i and (V, E) defines a connected graph.

To avoid ambiguity, we will sometimes use subscripts $(V_G, H_G, \ldots, i_G)$ when referring a specific graph G. A graph is called stable if all vertices satisfy the stability condition $2g(v) - 2 + n(v) > 0$, where $n\colon V \to \mathbb{Z}_{\geq 0}$ is the a function which assigns a vertex v to the total number of half edges in H incident to it. The total genus of a graph G is

$$\begin{aligned} g(G) :&= \sum_{v \in V} g(v) + h^1(G) \\ &= \sum_{v \in V} g(v) + |E| - |V| + 1. \end{aligned}$$

Any pointed stable curve C has an associated stable graph, called the *dual graph* of C, that encodes its topological data. The dual graph is constructed with the following rules: irreducible components of C correspond to vertices V, nodes of C correspond to edges E, and labeled marked points correspond to labeled half-edges N. Conversely, given any stable graph G, we define σ_G to be the closure of the locus of curves in $\overline{\mathcal{M}}_{g(G),|N|}$ whose dual graph is G. Another way to define σ_G is using the composition of gluing morphisms:

$$\iota_G\colon \prod_{v \in V} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n}, \tag{1}$$

where $(g, n) = (g(G), |N|)$, and points on curves are identified in the manner prescribed by the graph G ([GP, Proposition 8]). The boundary stratum σ_G is then equal to

$$\sigma_G := \mathrm{im}(\iota_G) \tag{2}$$

The loci of curves with a fixed dual graph form the stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type.

A *specialization* of a graph G is a graph H which is obtained by replacing each vertex of G with a graph of genus $g(v)$ with $n(v)$ half edges that are identified with the half edges incident to v. This corresponds roughly to specialization of curves.

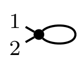
Definition 1. A G-structure on A is an identification of a specialization of G with A. In other words, it is a triple

$$(\alpha\colon V_A \twoheadrightarrow V_G, \beta\colon H_G \hookrightarrow H_A, \gamma\colon H_A \setminus \operatorname{im}(\beta) \to V_G)$$

which satisfies:

(1) The map β commutes with involution ($\beta \circ \iota_G = \iota_A \circ \beta$) and induces an isomorphism between the fixed points $N_G \xrightarrow{\sim} N_A$,
(2) Any half-edge $h \in \operatorname{im}(\beta)$ is incident to v if and only if $\beta^{-1}(h)$ is incident to $\alpha(v)$
(3) If $h \in H_A \setminus \operatorname{im}(B)$ is incident to v, then $\gamma(h) = \alpha(v)$.
(4) If $v \in V_G$, then the preimage $(\alpha^{-1}(v), \gamma^{-1}(v)/\iota_A)$ is a connected graph of genus $g(v)$.

Example 1. There can be many G-structures on the same graph A. Let G and A be the graphs denoted in the pictures below

$G =$  $A =$ 

There are four G-structures on A that respect the labels of the half-edges. The map β identifies one of the two edges of A with the edge of G in one of two different ways.

A (G, H)-graph is a graph A which has both a G-structure and an H-structure, called a (G, H)-structure. Two (G, H)-structures on a graph A are considered isomorphic if they differ by an automorphism of A. If A has a (G, H)-structure and $e = (h_1, h_2)$ is an edge of A, then we say e is a *common* (G, H)*-edge* if it is identified with both an edge of G and with an edge of H, i.e., if

$$h_1, h_2 \in \beta(H_G) \cap \beta(H_H) \subseteq H_A. \tag{3}$$

A (G, H)-graph A is called *generic* if every edge of A is identified with an edge of G, an edge of H, or both, i.e. if

$$\beta(H_G) \cup \beta(H_H) = H_A. \tag{4}$$

The set of all generic (G, H)-structures is denoted $\Gamma(G, H)$.

Example 2. Let A and G be the same graphs as in Example 1, and set $H = G$. There are sixteen (G, H)-structures on A. Eight of them are generic. They are isomorphic in pairs; i.e., up to an automorphism of A there are only eight different (G, H)-structures, four of which are generic.

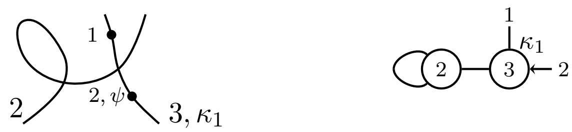


FIGURE 1. A 2-pointed curve of genus 6 and its dual decorated graph

3. Decorated graphs

A *decorated stable graph* G is a stable graph $\overline{G}$ with the additional data of a monomial

$$\theta_v = \prod_{i=1}^{n(v)} \psi_i^{e_i} \prod_{j=1}^{m} \kappa_j^{f_j} \tag{5}$$

chosen for each vertex v. Define σ_G to be the tautological class

$$\sigma_G := \frac{1}{|\mathrm{Aut}(G)|} \iota_{\overline{G}*} \left(\prod_{v \in V} \theta_v \right). \tag{6}$$

Here $\overline{G}$ denotes the underlying stable graph of G without any vertex decorations.

Remark 1. The cotangent ψ-classes are indexed by the half-edges incident to a vertex, so we denote them by adding arrowheads to the appropriate half-edge (Figure 1).

Let $\Sigma^*(\overline{\mathcal{M}}_{g,n})$ denote the graded vector space of decorated genus g stable graphs with n labeled half-edges, with the grading by codimension

$$\mathrm{codim}(G) := |E_G| + \sum_{v \in V} \mathrm{codim}(\theta_v), \tag{7}$$

where

$$\mathrm{codim} \left(\prod_{i=1}^{n} \psi_i^{e_i} \prod_{j=1}^{m} \kappa_j^{f_j} \right) = \sum_{i=1}^{n} e_i + \sum_{j=1}^{m} j f_j. \tag{8}$$

For any decorated graph A, and vertex $v \in V_A$, let $\kappa_{a,v} \cdot A$ denote the graph where the decoration θ_v is replaced with $\kappa_a \theta_v$, with associative and distributive rules

$$(\kappa_{a,v} \kappa_{a',v}) \cdot A = \kappa_{a,v} \cdot (\kappa_{a',v} \cdot A), \tag{9}$$

$$(\kappa_{a,v} + \kappa_{a',v}) \cdot A = \kappa_{a,v} \cdot A + \kappa_{a',v} \cdot A. \tag{10}$$

Similarly, if h is any half-edge of A, then we let $\psi_h \cdot A$ denote the graph A where the half-edge h is decorated with an additional arrowhead.

Example 3. The pullback of the class κ_a to a boundary stratum B is the sum of strata

$$\kappa_a \sigma_A = \sum_{v \in V(A)} \sigma_{\kappa_{a,v} \cdot A}. \tag{11}$$

Let G be a decorated graph, and let A be an undecorated graph with a $\overline{G}$-structure (α, β, γ). Let v be a vertex in G decorated with the monomial $\theta_v := \prod_{i=1}^n \psi_i^{e_i} \prod_{j=1}^m \kappa_j^{f_j}$, and set

$$f_A(G, v) = \prod_{i=1}^n \psi_{\beta(i)}^{e_i} \prod_{j=1}^m \left(\sum_{w \in \alpha^{-1}(v)} \kappa_{j,w} \right)^{f_j}, \tag{12}$$

$$F_A(G, H) = \sum_{v \in V_G} f_A(G, v) \sum_{v \in V_H} f_A(H, v) \prod_{e=(h_1,h_2)} (-\psi_{h_1} - \psi_{h_2}). \tag{13}$$

The last product is taken over all common $(\overline{G}, \overline{H})$-edges of A.

We define multiplication in $\Sigma^*(\overline{\mathcal{M}}_{g,n})$ with the formula

$$G \cdot H := \sum_{A \in \Gamma(\overline{G}, \overline{H})} \frac{1}{|\mathrm{Aut}(A)|} F_A(G, H) \cdot A, \tag{14}$$

thus turning Σ^* into a graded algebra.

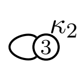
Proposition 1. *There is a natural surjective map of rings*

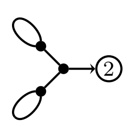
$$\Phi \colon \Sigma^*(\overline{\mathcal{M}}_{g,n}) \to R^*(\overline{\mathcal{M}}_{g,n}) \tag{15}$$

which sends a decorated dual graph to its associated stratum class.

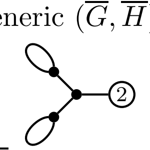
Proof. This is a map from [GP, Formula 11], and surjectivity follows from [GP, Proposition 11]. $\square$

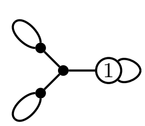
Example 4. We calculate the product of σ_G and σ_H, where G and H are the following decorated stable graphs in $\Sigma^3(\overline{\mathcal{M}}_{4,0})$ and $\Sigma^6(\overline{\mathcal{M}}_{4,0})$. Note that vertices of genus 0 are denoted by dots.

$G =$ 

$H =$ 

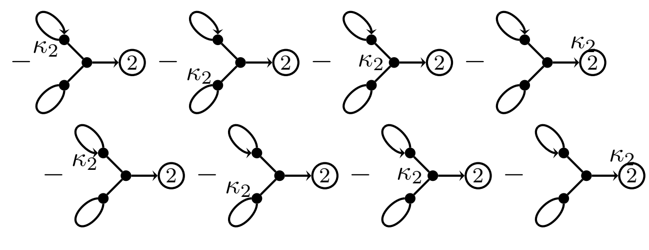
There are two generic $(\overline{G}, \overline{H})$-graphs:

$A =$ 

$B =$ 

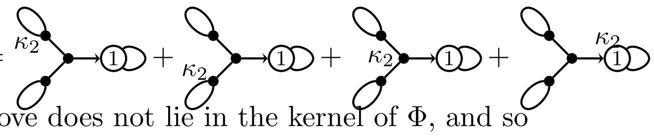
Graph A has four $\overline{G}$-structures and eight $\overline{H}$-structures, a total of 32 $(\overline{G}, \overline{H})$-structures, each of which has exactly one common edge. The order of the automorphism group of A is eight. Graph B has sixteen generic $(\overline{G}, \overline{H})$-structure with no common edges. The order of the automorphism group of B is sixteen.

According to formula (14), the product $F_A(G,H) \cdot A$ is the sum of eight graphs

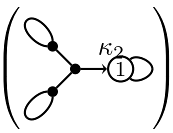
(16) $F_A(G,H) \cdot A :=$ 

all of which lie in the kernel of Φ.

The product $F_B(G,H) \cdot B$ is the sum of four graphs

(17) $F_B(G,H) \cdot B :=$ 

Only the last term above does not lie in the kernel of Φ, and so

(18) $\sigma_G \cdot \sigma_H = \Phi(G \cdot H) = \Phi\Bigg($  $\Bigg) = \int_{\overline{\mathcal{M}}_{1,3}} \psi_1 \kappa_2 = \frac{1}{8}.$

4. Curves of compact type, genus 4 and 5

Recall from §1 that $\mathcal{M}_{g,n}^{\mathrm{ct}}$ denotes the locus of curves in $\overline{\mathcal{M}}_{g,n}$ of compact type and $\mathcal{M}_{g,n}^{\mathrm{rt}}$ denotes the locus of curves in $\overline{\mathcal{M}}_{g,n}$ with rational tails. Define the tautological rings $R^*(\mathcal{M}_{g,n}^{\mathrm{ct}})$ and $R^*(\mathcal{M}_{g,n}^{\mathrm{rt}})$ by restriction. Conjecturally, the rings $R^*(\overline{\mathcal{M}}_{g,n})$, respectively $R^*(\mathcal{M}_{g,n}^{\mathrm{ct}})$ and $R^*(\mathcal{M}_{g,n}^{\mathrm{rt}})$ are Gorenstein with top degree $3g-3+n$, respectively $2g-3+n$ and $g-2+n$ ([AC, HKK$^+$, M]). The one-dimensionality of the top degrees are known in all cases ([L, F2, GV]), with the isomorphism given by evaluation

$$\alpha \mapsto \int_{\overline{\mathcal{M}}_{g,n}} \alpha, \tag{19}$$

in the case of $\overline{\mathcal{M}}_{g,n}$, and using the evaluation classes λ_g and $\lambda_g \lambda_{g-1}$ in the cases of $\mathcal{M}_{g,n}^{\mathrm{ct}}$ and $\mathcal{M}_{g,n}^{\mathrm{rt}}$,

$$R^{2g-3+n}(\mathcal{M}_{g,n}^{\mathrm{ct}}) \cong \mathbb{Q} \qquad\qquad R^{g-2+n}(\mathcal{M}_{g,n}^{\mathrm{rt}}) \cong \mathbb{Q} \tag{20}$$

$$\alpha \mapsto \int_{\overline{\mathcal{M}}_{g,n}} \alpha\lambda_g \qquad\qquad \alpha \mapsto \int_{\overline{\mathcal{M}}_{g,n}} \alpha\lambda_g\lambda_{g-1}. \tag{21}$$

In this section we show that $R^*(\mathcal{M}_4^{\mathrm{ct}})$ is in fact Gorenstein, and discuss the structure of $R^*(\mathcal{M}_5^{\mathrm{ct}})$.

By [FP, Proposition 2], any tautological decoration θ_v of sufficiently high codimension lies on the tautological ring of the boundary $R^*(\partial\overline{\mathcal{M}}_{g,n})$, defined to be the subring of $A^*(\partial\overline{\mathcal{M}}_{g,n})$ generated by pushforwards of tautological classes on boundary divisors via gluing morphisms. As a result, the

map Φ defined in Proposition 1 remains surjective when restricted to the graphs whose vertex decorations satisfy

$$\operatorname{codim}(\theta_v) < g(v) + \delta_{0g(v)} - \delta_{0n(v)}, \tag{22}$$

and the tautological restriction sequence

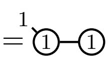
$$R^{k-1}(\partial\overline{\mathcal{M}}_{g,n}) \longrightarrow R^k(\overline{\mathcal{M}}_{g,n}) \longrightarrow R^k(\mathcal{M}_{g,n}) \longrightarrow 0 \tag{23}$$

is exact in degrees $k \geq g(v) + \delta_{0g(v)} - \delta_{0n(v)}$. Note that since $A^k(\partial\overline{\mathcal{M}}_{g,n}) = R^k(\partial\overline{\mathcal{M}}_{g,n})$ when $k = 0, 1$, the restriction sequence is also exact when $k = 1, 2$. The exactness of this sequence for intermediate values remains unknown ([FP, Conjecture 2]).

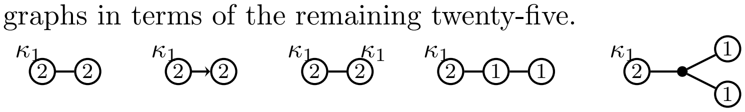
There are 30 graphs of genus 4 whose decorations satisfy inequality (22). Five relations among the thirty strata represented by these graphs can be seen as as follows: the relation $\kappa_1 = 0$ on $\mathcal{M}_2$ pulls back to $\kappa_1 - \psi_1 = 0$ on $\mathcal{M}_{2,1}$. By exactness of the restriction sequence (23) when $k = 1$, this extends to a relation on $\mathcal{M}_{2,1}$,

$$\kappa_1 - \psi_1 = \frac{7}{5}\delta_G \tag{24}$$

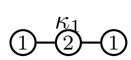
where G is the graph

(25) $G =$ 

This relation, when pushed forward via various gluing morphisms to relations on $\mathcal{M}_4^{\mathrm{ct}}$, allows one to express the strata corresponding to the following five graphs in terms of the remaining twenty-five.

(26) 

Similarly, relation (24) can be pulled back to $\mathcal{M}_{2,2}^{\mathrm{ct}}$ and then pushed forward via gluing morphisms, allowing us to eliminate one more graph:

(27) 

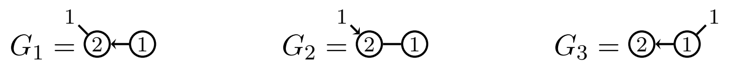
In a similar manner we get three additional relations from $R^2(\mathcal{M}_{3,1})$, which is one-dimensional. (This is the socle statement of Faber's conjecture, which is proved in [FP, Proposition 3] and [GV, §5.8].) Since (23) is exact when $k = 2$, the relations $\psi_1\kappa_1 = 5\psi_1^2$, $\kappa_2 = \psi_1^2$, and $\kappa_1^2 = 9\psi_1^2$ extend to relations on $\mathcal{M}_{3,1}^{\mathrm{ct}}$

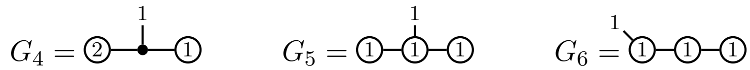
(28)
$$\psi_1\kappa_1 = 5\psi_1^2 + \frac{16}{21}\sigma_{G_1} + \frac{5}{7}\sigma_{G_2} + \frac{40}{21}\sigma_{G_3} - \frac{61}{21}\sigma_{G_4} + \frac{4}{35}\sigma_{G_5} - \frac{16}{35}\sigma_{G_6}$$

$$\kappa_2 = \psi_1^2 + \frac{41}{21}\sigma_{G_1} + \frac{41}{21}\sigma_{G_3} - \frac{41}{21}\sigma_{G_4} - \frac{4}{35}\sigma_{G_5} - \frac{8}{35}\sigma_{G_6} \tag{29}$$

$$\kappa_1^2 = 9\psi_1^2 + \frac{299}{21}\sigma_{G_1} + \frac{10}{7}\sigma_{G_2} + \frac{347}{21}\sigma_{G_3} - \frac{389}{21}\sigma_{G_4} + \frac{19}{35}\sigma_{G_5} - \frac{2}{7}\sigma_{G_6} \tag{30}$$

where

(31) 

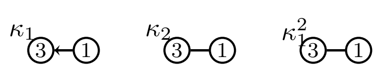
(32) 

The coefficients above were found using a Maple program which implemented the algorithm described in the previous section. Specifically, this program calculated the 7 by 10 matrix of intersection numbers below

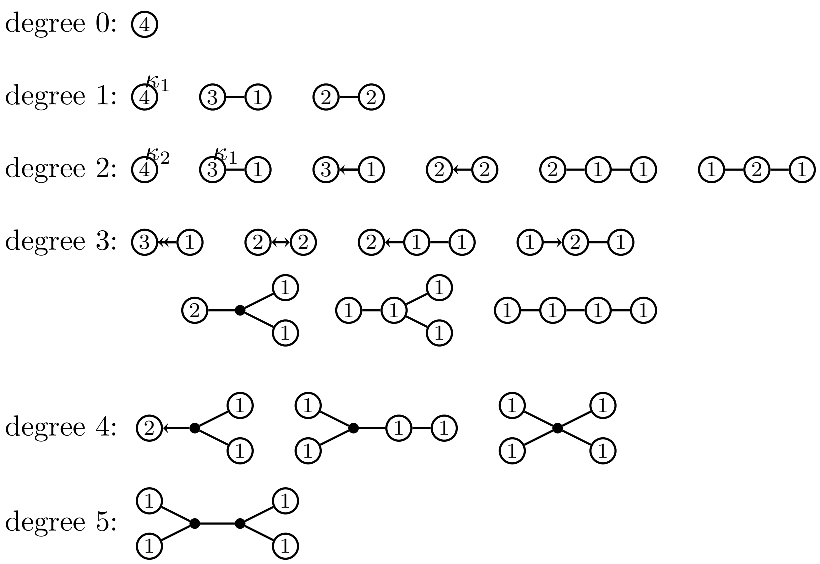
	ψ_1^2	σ_{G_1}	σ_{G_2}	σ_{G_3}	σ_{G_4}	σ_{G_5}	σ_{G_6}	$\psi_1\kappa_1$	κ_2	κ_1^2
ψ_1^2	$\frac{31}{70}$	$\frac{21}{10}$	$\frac{7}{10}$	0	0	1	0	$\frac{31}{7}$	$\frac{31}{7}$	$\frac{248}{7}$
σ_{G_1}	$\frac{21}{10}$	$\frac{13}{10}$	$\frac{-1}{10}$	0	1	-1	0	$\frac{21}{5}$	$\frac{14}{5}$	$\frac{91}{5}$
σ_{G_2}	$\frac{7}{10}$	$\frac{-1}{10}$	$\frac{-11}{10}$	1	0	-3	0	$\frac{42}{5}$	$\frac{14}{5}$	$\frac{91}{5}$
σ_{G_3}	0	0	1	$\frac{-7}{10}$	$\frac{-7}{10}$	2	-1	$\frac{14}{5}$	0	$\frac{21}{5}$
σ_{G_4}	0	1	0	$\frac{-7}{10}$	0	1	-1	$\frac{21}{10}$	$\frac{7}{10}$	$\frac{7}{2}$
σ_{G_5}	1	-1	-3	2	1	0	0	3	1	5
σ_{G_6}	0	0	0	-1	-1	0	0	2	0	2

where each entry is 13824 times the intersection number of the classes indexing the rows and columns.

The relations above span its 3-dimension kernel. Because of these three relations, we may also disregard graphs in $\mathcal{M}_4^{\mathrm{ct}}$ whose genus 3 vertices are decorated with $\kappa_1\psi_1$, κ_2, and κ_1^2.

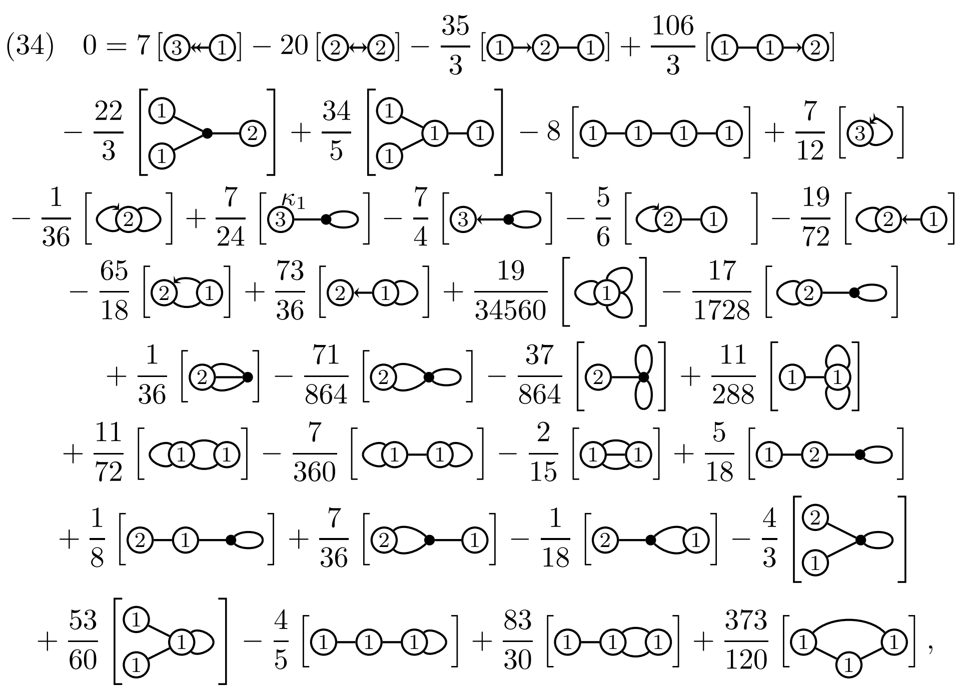
(33) 

The remaining 21 decorated dual graphs are listed below.



Note that there are six decorated strata classes in degree 2, and seven in degree 3. With no other obvious dependencies among these classes, the conjectured Gorenstein condition for $R^*(\mathcal{M}_g^{\mathrm{ct}})$ suggests the existence of a new degree 3 relation ([P]).

Proposition 2. *The following new relation holds among classes in $H_{12}(\overline{\mathcal{M}}_4)$.*



where $[G]$ denotes the tautological class σ_G.

This relation was shown to hold in ring $R^3(\overline{\mathcal{M}}_4)$ by Faber and Pandharipande, by a different method. Their work is not yet published.

Proof. The cohomology of $\mathcal{M}_4$ and $\overline{\mathcal{M}}_4$, calculated respectively by Tommasi and Bergström-Tommasi in [T] and [BT], is known to be isomorphic to cohomology with compact support. Dual to the short exact sequence

$$0 \to H_c^{12}(\overline{\mathcal{M}}_4) \to H_c^{12}(\partial\mathcal{M}_4) \to H_c^{13}(\mathcal{M}_4) \to 0 \tag{35}$$

is a sequence of Borel-Moore homology groups

$$0 \to H_{13}^{BM}(\mathcal{M}_4) \to H_{12}^{BM}(\partial\mathcal{M}_4) \to H_{12}^{BM}(\overline{\mathcal{M}}_4) \to 0. \tag{36}$$

The first term $H_{13}^{BM}(\mathcal{M}_4)$ is isomorphic to $H^5(\mathcal{M}_4)$ and is known to be 1-dimensional ([T]). The second term is isomorphic to $H_{12}(\partial\mathcal{M}_4)$ and is spanned by the 33 terms in equation (34). The kernel of the map ϕ in the equivalent sequence

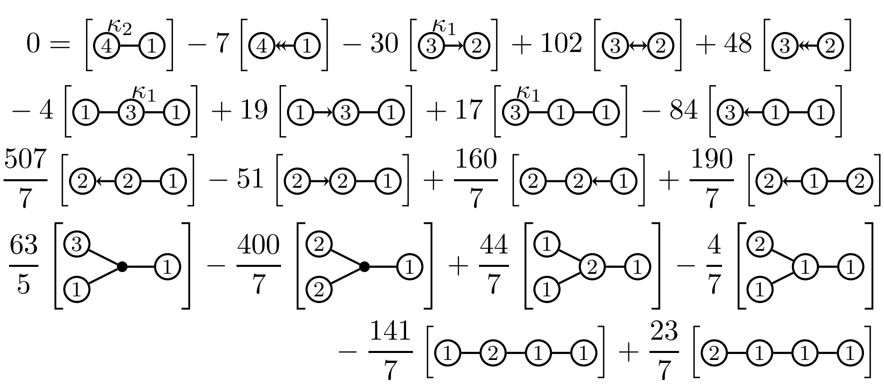
$$0 \to H^5(\mathcal{M}_4) \to H_{12}(\partial\mathcal{M}_4) \xrightarrow{\phi} H_{12}(\overline{\mathcal{M}}_4) \to 0 \tag{37}$$

is exactly the relation given above.

□

A similar analysis can be done for $R^3(\mathcal{M}_5^{\mathrm{ct}})$. There are 31 graphs in $\Sigma^3(\mathcal{M}_5^{\mathrm{ct}})$, eleven of which can be eliminated using relations from genus 2, 3, and 4. The rank of the intersection pairing $R^3 \times R^4$ on the remaining 20 graphs is only 19, which suggests a new codimension 3 relation in $\mathcal{M}_5^{\mathrm{ct}}$.

Conjecture 1. *The following relation holds in $R^3(\mathcal{M}_5^{\mathrm{ct}})$.*

(38)


5. Gorenstein quotients of tautological rings

Below we display the ranks of the intersection pairing on $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}_{g,n}^{\mathrm{ct}}$, and $\mathcal{M}_{g,n}^{\mathrm{rt}}$, respectively. These were calculated using Proposition 1, where Φ is restricted to the finite set of graphs satisfying inequality (22).

Rank of the intersection pairing on $\overline{\mathcal{M}}_{g,n}$

	Codimension									
(g,n)	0	1	2	3	4	5	6	7	8	9
$(0,3)$	1									
$(0,4)$	1	1								
$(0,5)$	1	5	1							
$(0,6)$	1	16	16	1						
$(0,7)$	1	42	127	42	1					
$(1,1)$	1	1								
$(1,2)$	1	2	1							
$(1,3)$	1	5	5	1						
$(1,4)$	1	12	23	12	1					
$(1,5)$	1	27	102	102	27	1				
$(2,0)$	1	2	2	1						
$(2,1)$	1	3	5	3	1					
$(2,2)$	1	6	14	14	6	1				
$(2,3)$	1	12	44	67	44	12	1			
$(2,4)$	1	24	144	333	333	144	24	1		
$(3,0)$	1	3	7	10	7	3	1			
$(3,1)$	1	5	16	29	29	16	5	1		
$(3,2)$	1	9	42	104	142	104	42	9	1	
$(4,0)$	1	4	13	32	50	50	32	13	4	1

Rank of the intersection pairing on $\mathcal{M}_{g,n}^{\mathrm{ct}}$

	Codimension							
(g,n)	0	1	2	3	4	5	6	7
$(1,1)$	1							
$(1,2)$	1	1						
$(1,3)$	1	4	1					
$(1,4)$	1	11	11	1				
$(1,5)$	1	26	71	26	1			
$(1,6)$	1	57	348	348	57	1		
$(2,0)$	1	1						
$(2,1)$	1	2	1					
$(2,2)$	1	5	5	1				
$(2,3)$	1	11	24	11	1			
$(2,4)$	1	23	101	101	23	1		
$(2,5)$	1	47	384	769	384	47	1	
$(3,0)$	1	2	2	1				
$(3,1)$	1	4	7	4	1			
$(3,2)$	1	8	24	24	8	1		
$(3,3)$	1	16	82	144	82	16	1	
$(3,4)$	1	32	274	813	813	274	32	1
$(4,0)$	1	3	6	6	3	1		
$(4,1)$	1	5	17	25	17	5	1	
$(4,2)$	1	10	51	120	120	51	10	1
$(5,0)$	1	3	10	19	19	10	3	1

Rank of the intersection pairing on $\mathcal{M}^{\mathrm{rt}}_{g,n}$

	Degree								
(g,n)	0	1	2	3	4	5	6	7	8
$(2,0)$	1								
$(2,1)$	1	1							
$(2,2)$	1	3	1						
$(2,3)$	1	7	7	1					
$(2,4)$	1	15	35	15	1				
$(2,5)$	1	31	147	147	31	1			
$(2,6)$	1	63	556	1126	556	63	1		
$(3,0)$	1	1							
$(3,1)$	1	2	1						
$(3,2)$	1	4	4	1					
$(3,3)$	1	8	15	8	1				
$(3,4)$	1	16	54	54	16	1			
$(3,5)$	1	32	188	333	188	32	1		
$(4,0)$	1	1	1						
$(4,1)$	1	2	2	1					
$(4,2)$	1	4	6	4	1				
$(4,3)$	1	8	19	19	8	1			
$(4,4)$	1	16	61	95	61	16	1		
$(4,5)$	1	32	199	470	470	199	32	1	
$(5,0)$	1	1	1	1					
$(5,1)$	1	2	3	2	1				
$(5,2)$	1	4	8	8	4	1			
$(5,3)$	1	8	22	33	22	8	1		
$(5,4)$	1	16	65	136	136	65	16	1	
$(5,5)$	1	32	204	577	852	577	204	32	1
$(6,0)$	1	1	2	1	1				
$(6,1)$	1	2	4	4	2	1			
$(6,2)$	1	4	9	13	9	4	1		
$(6,3)$	1	8	23	44	44	23	8	1	
$(6,4)$	1	16	66	159	226	159	66	16	1
$(7,0)$	1	1	2	2	1	1			
$(7,1)$	1	2	4	5	4	2	1		

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